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Retailer's optimal pricing and lot-sizing policies for deteriorating items with partial backlogging

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Abstract

Pricing is a major strategy for a retailer to obtain its maximum profit. Therefore, in this paper, we establish an economic order quantity model for a retailer to determine its optimal selling price, replenishment number and replenishment schedule with partial backlogging. We first prove that the optimal replenishment schedule not only exists but also is unique, for any given selling price. Next, we show that the total profit is a concave function of p when the replenishment number and schedule are given. We then provide a simple algorithm to find the optimal selling price, replenishment number and replenishment timing for the proposed model. Finally, we use a couple of numerical examples to illustrate the algorithm.

Keywords: Pricing; Lot-sizing; Deteriorating items; Partial backlogging

1. Introduction

In the growth stage of a product life cycle, the demand rate can be well approximated by a linear form. Consequently, Resh et al. [16], and Donaldson [8] established an algorithm to determine the optimal replenishment number and timing for a linearly increasing demand pattern. Barbosa and Friedman [4] then generalized the solutions for power-form demand models. Furthermore, Henery [14] extended the demand to

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any log-concave demand function. Following the approach of Donaldson, Dave [6] developed an exact replenishment policy for an inventory model with shortages. In contrast to the traditional replenishment policy that does not start with shortages, Goyal et al. [11] proposed an alternative that starts with shortages in every cycle, and suggested that their alternative outperforms the traditional approach. Lately, Teng et al. [21] investigated various inventory replenishment models with shortages, and mathematically proved that the alternative by Goyal et al. is, indeed, less expensive to operate than the traditional policy. Hariga and Goyal [13] then developed an iterative procedure that is simpler than that by Goyal et al. Later, Teng [19] proposed a simple and computationally efficient optimal method in recursive fashion to solve the problem.

In many real-life situations, products deteriorate continuously such as medicine, volatile liquids, blood banks, and others. Consequently, Dave and Patel [7] discussed an inventory model for deteriorating items when shortages were not allowed. Sachan [17] then extended their model to allow for shortages. Hariga [12] developed optimal EOQ models with log-concave demand for deteriorating items under three replenishment policies. Recently, Yang et al. [23] provided various inventory models with time-varying demand patterns under inflation. Lately, Goyal and Giri [10] reviewed the contributions on the literature in modeling of deteriorating inventory.

The characteristic of all of the above articles is that the unsatisfied demand (due to shortages) is completely backlogged. However, for fashionable commodities and high-tech products with short product life cycle, the willingness for a customer to wait for backlogging during a shortage period is diminishing with the length of the waiting time. Hence, the longer the waiting time is, the smaller the backlogging rate would be. To reflect this phenomenon, Abad [1] proposed several pioneer and inspiring backlogging rates to be decreasing functions of waiting time. Lately, he [2] added the pricing strategy into consideration, and provided the optimal price and lot-size for a reseller when demand was a known function of the selling price. Chang and Dye [5] then developed a finite-horizon inventory model by using Abad's reciprocal backlogging rate. Concurrently, Papachristos and Skouri [15] established a multi-period inventory model based on Abad's negative exponential backlogging rate. Recently, Teng et al. [20] extended the fraction of unsatisfied

Table 1
Summary of the related research

Author	Demand factors	Demand patterns	Deterioration	Backlogging	Planning horizon
Abad [1,2]	Price	Constant	Yes	Time-dependent backlogging	Infinite
Resh et al. [16]	Time	Linear	No	No	Finite
Donaldson [8]	Time	Linear	No	No	Finite
Barbosa and Friedman [4]	Time	Power-form	No	No	Finite
Henery [14]	Time	Log-concave	No	No	Finite
Dave [6]	Time	Linear	No	Complete backlogging	Finite
Goyal et al. [11]	Time	Linear	No	Complete backlogging	Finite
Teng et al. [21]	Time	Linear	No	Complete backlogging	Finite
Hariga and Goyal [13]	Time	Linear	No	Complete backlogging	Finite
Teng [19]	Time	Linear	No	Complete backlogging	Finite
Dave and Patel [7]	Time	Linear	Yes	No	Finite
Sachan [17]	Time	Linear	Yes	Complete backlogging	Finite
Hariga [12]	Time	Log-concave	Yes	Complete backlogging	Finite
Yang et al. [23]	Time	Log-concave	Yes	Complete backlogging	Finite
Teng et al. [22]	Time	Log-concave	Yes	Complete backlogging	Finite
Chang and Dye [5]	Time	Log-concave	Yes	Time-dependent backlogging	Finite
Papachristos and Skouri [15]	Time	Log-concave	Yes	Time-dependent backlogging	Finite
Teng et al. [20]	Time	Log-concave	Yes	Time-dependent backlogging	Finite
Wee [24]	Time and price	Exponential	Yes	Fixed fraction of shortages	Finite
Present paper	Time and price	Log-concave	Yes	Time-dependent backlogging	Finite

demand backordered in both models [5,15] to any decreasing function of the waiting time up to the next replenishment. As a result, our model included numerous previous models, such as [4–9,11–17,19–22], as special cases.

Pricing is a major strategy for a seller to achieve its maximum profit. Consequently, in this paper, we extend our previous model [20] to provide a retailer its optimal selling price, replenishment number and replenishment schedule simultaneously in order to achieve its maximum profit. The major assumptions used in related research are summarized in Table 1. The rest of the paper is organized as follows. In Section 2, we describe the assumptions and notation used throughout this study. In Section 3, we establish the mathematical model and the necessary conditions for finding an optimal solution. We then prove that the optimal replenishment schedule not only exists but also is unique, for any given selling price. Furthermore, we show that the total profit is a concave function of selling price when the replenishment number and schedule are given. In Section 4, we provide a simple algorithm to find the optimal replenishment schedule and selling price for the proposed model. In Section 5, we use a couple of numerical examples to illustrate the algorithm. Finally, we make a summary and provide some suggestions for future research in Section 6.

2. Assumptions and notation

2.1. Assumptions

The mathematical model of the inventory replenishment problem is based on the following assumptions:

1. The planning horizon of the inventory problem here is finite and is taken as H time units.
2. Lead time is zero.
3. The initial inventory level is zero.
4. A constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory.
5. Shortages are allowed. The fraction of shortages backordered is a decreasing function $\beta(x)$, where x is the waiting time up to the next replenishment, and $0 \leqslant \beta(x) \leqslant 1$ with $\beta(0)=1$. To guarantee the existence of an optimal solution, we assume that $\beta(x)+H\beta'(x)>0$, where $\beta'(x)$ is the first derivative of $\beta(x)$. Note that if $\beta(x)=1$ (or 0) for all x, then shortages are completely backlogged (or lost).
6. In today's global competition, many firms have no pricing power. As a result, the selling price is hardly changed for many firms. Therefore, in today's global competition and low inflation environment, we may assume WLOG that the selling price is constant within a couple of years. In a second paper, we will discuss the case in which the selling price is changed from one cycle to another.

2.2. Notations

θ	the deterioration rate
c_{f}	the fixed purchasing cost per order
p	the selling price per unit (a decision variable), defined in the interval $[0,p_u]$, where p_u is a very large number
$f(t,p)$	the demand rate at time t and price p with $f(t,p)=g(t)A(p)$, where $g(t)$ is positive and log-concave in the planning horizon $(0,H]$ and $A(p)$ is any non-negative, continuous, convex, decreasing function of the selling price in $[0,p_u]$
c_{v}	the variable purchasing cost per unit
c_{h}	the inventory holding cost per unit per unit time

c_s the backlogging cost per unit per unit time due to shortages
c_l the unit cost of lost sales
n the number of replenishments over $[0, H]$ (a decision variable)
t_i the ith replenishment time (a decision variable), $i = 1, 2, \ldots, n$
s_i the time at which the inventory level reaches zero in the ith replenishment cycle (a decision variable), $i = 1, 2, \ldots, n$

As a result, the decision problem here has $2n+2$ decision variables.

3. Model formulation

For simplicity, we use the same inventory shortage model as in Teng et al. [20], which is shown in Fig. 1. Similarly, we obtain the time-weighted inventory during the ith replenishment cycle as

$$I_i = \frac{1}{\theta}\int_{t_i}^{s_i} [\mathrm{e}^{\theta(t-t_i)} - 1] f(t,p)\,\mathrm{d}t, \quad i = 1, 2, \ldots, n, \tag{1}$$

the time-weighted backorders due to shortages during the ith cycle as

$$S_i = \int_{s_{i-1}}^{t_i} (t_i - t)\beta(t_i - t) f(t,p)\,\mathrm{d}t \tag{2}$$

the number of lost sales during the ith cycle as

$$L_i = \int_{s_{i-1}}^{t_i} [1 - \beta(t_i - t)] f(t,p)\,\mathrm{d}t \tag{3}$$

the order quantity at t_i, in the ith replenishment cycle as

$$Q_i = \int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t,p)\,\mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t,p)\,\mathrm{d}t, \quad i = 1, 2, \ldots, n, \tag{4}$$

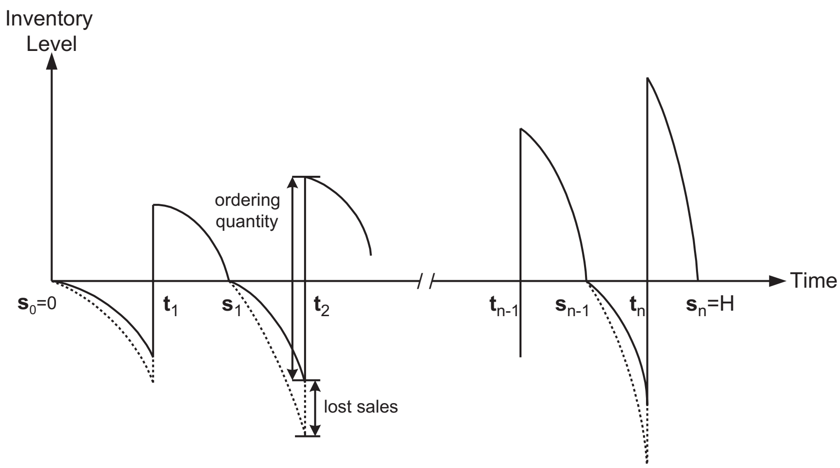

Fig. 1. Graphical representation of inventory system.

the purchase cost during the ith replenishment cycle as

$$P_i = c_{\mathrm{f}} + c_{\mathrm{v}}Q_i = c_{\mathrm{f}} + c_{\mathrm{v}}\left[\int_{s_{i-1}}^{t_i} \beta(t_i - t)f(t,p)\,\mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)}f(t,p)\,\mathrm{d}t\right], \quad i = 1,2,\ldots,n, \tag{5}$$

and the number of units sold in the ith replenishment cycle as

$$R_i = \int_{s_{i-1}}^{t_i} \beta(t_i - t)f(t,p)\,\mathrm{d}t + \int_{t_i}^{s_i} f(t,p)\,\mathrm{d}t, \quad i = 1,2,\ldots,n. \tag{6}$$

Hence, if n replenishment orders are placed in $[0,H]$, then the total profit of the inventory system during the planning horizon H is

$$\begin{aligned}\mathrm{TP}(p,n,\{s_i\},\{t_i\}) &= \sum_{i=1}^{n}(pR_i - P_i - c_{\mathrm{h}}I_i - c_{\mathrm{s}}S_i - c_{\mathrm{l}}L_i)\\ &= (p-c_{\mathrm{v}})\sum_{i=1}^{n}\left[\int_{s_{i-1}}^{t_i}\beta(t_i-t)f(t,p)\,\mathrm{d}t + \int_{t_i}^{s_i}f(t,p)\,\mathrm{d}t\right]\\ &\quad - nc_{\mathrm{f}} - \frac{c_{\mathrm{h}}+c_{\mathrm{v}}\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}[\mathrm{e}^{\theta(t-t_i)}-1]f(t,p)\,\mathrm{d}t - c_{\mathrm{s}}\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}(t_i-t)\beta(t_i-t)f(t,p)\,\mathrm{d}t\\ &\quad - c_{\mathrm{l}}\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}[1-\beta(t_i-t)]f(t,p)\,\mathrm{d}t.\end{aligned} \tag{7}$$

Now, the problem is to determine $p, n, \{s_i\}$ and $\{t_i\}$ such that $\mathrm{TP}(p,n,\{s_i\},\{t_i\})$ is maximized. Hence, it is a $2+2n$ decision-making problem for a retailer.

4. Optimal solution and theoretical results

For any given n and p, the necessary conditions for $\mathrm{TP}(\{s_i\},\{t_i\}\,|\,n, p)$ to be maximum are

$$\begin{aligned}\frac{\partial\mathrm{TP}(\{s_i\},\{t_i\}\mid n,p)}{\partial s_i} &= (p-c_{\mathrm{v}}+c_{\mathrm{l}})[1-\beta(t_{i+1}-s_i)]f(s_i,p)\\ &\quad - \frac{c_{\mathrm{h}}+c_{\mathrm{v}}\theta}{\theta}[\mathrm{e}^{\theta(s_i-t_i)}-1]f(s_i,p) + c_{\mathrm{s}}(t_{i+1}-s_i)\beta(t_{i+1}-s_i)f(s_i,p) = 0\end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\partial\mathrm{TP}(\{s_i\},\{t_i\}\mid n,p)}{\partial t_i} &= (p-c_{\mathrm{v}}+c_{\mathrm{l}})\int_{s_{i-1}}^{t_i}\beta'(t_i-t)f(t,p)\,\mathrm{d}t + (c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\int_{t_i}^{s_i}\mathrm{e}^{\theta(t-t_i)}f(t,p)\,\mathrm{d}t\\ &\quad - c_{\mathrm{s}}\left[\int_{s_{i-1}}^{t_i}[\beta(t_i-t)+(t_i-t)\beta'(t_i-t)]f(t,p)\,\mathrm{d}t\right] = 0.\end{aligned} \tag{9}$$

Using the fact that $f(t,p)=g(t)A(p)$, after rearranging the terms in (8) and (9), we thus get

$$\frac{c_{\mathrm{h}}+c_{\mathrm{v}}\theta}{\theta}[\mathrm{e}^{\theta(s_i-t_i)}-1] = c_{\mathrm{s}}(t_{i+1}-s_i)\beta(t_{i+1}-s_i) + (p-c_{\mathrm{v}}+c_{\mathrm{l}})[1-\beta(t_{i+1}-s_i)], \quad i = 1,2,\ldots,n-1, \tag{10}$$

and

$$(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\int_{t_i}^{s_i}\mathrm{e}^{\theta(t-t_i)}g(t)\,\mathrm{d}t=c_{\mathrm{s}}\int_{s_{i-1}}^{t_i}[\beta(t_i-t)+(t_i-t)\beta'(t_i-t)]g(t)\,\mathrm{d}t,$$
$$-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\int_{s_{i-1}}^{t_i}\beta'(t_i-t)g(t)\,\mathrm{d}t,\quad i=1,2,\ldots,n. \tag{11}$$

Applying (10) and (11), we obtain the following results.

Theorem 1. *For any given n and p, we have*

(a) *The system of equations (10) and (11) has a unique solution.*
(b) *The solution in (a) satisfies the second-order conditions for maximum.*

Proof. Please see Appendix A for details. □

Theorem 1 reduces the $2n$-dimensional problem of finding the optimal schedule $\{s_i^*\}$ and $\{t_i^*\}$ to a one-dimensional problem. Since $s_0^*=0$, we need only to find t_1^* to generate s_1^* by (11), t_2^* by (10), and then the rest of $\{t_i^*\}$ and $\{s_i^*\}$ uniquely by repeatedly using (10) and (11). For any chosen t_i^*, if $s_n^*=H$, then t_1^* is chosen correctly. Otherwise, we can easily find the optimal t_1^* by standard search schemes. The solution procedure for finding $\{t_i^*\}$ and $\{s_i^*\}$ can be obtained by the algorithm in Yang et al. [23] with $L=H/(4n)$ and $U=H/n$.

Next, we show that the total profit $\mathrm{TP}(p,n,\{s_i^*\},\{t_i^*\})$ for any given p is a concave function of the number of replenishments. As a result, the search for the optimal replenishment number, n^*, is reduced to find a local maximum. For simplicity, let

$$\mathrm{TP}(n\mid p)=\mathrm{TP}(p,n,\{s_i^*\},\{t_i^*\}\mid p). \tag{12}$$

We have the following theorem.

Theorem 2. *$TP(n|p)$ is concave in n.*

Proof. The technique used in the proof of this theorem involves dynamic programming arguments and is similar to that used by Teng et al. [22] or Friedman [9]. For further details, please see Teng et al. [22]. □

To avoid using a brute force enumeration for finding n^*, we further simplify the search process by providing an intuitively good starting value for n^*. In fact, the holding cost per unit (including inventory and deterioration costs) is $c_{\mathrm{h}}+\theta c_{\mathrm{v}}$. The unit penalty cost of lost sales (which is the sum of profit per unit and cost of lost goodwill) is $p-c_{\mathrm{v}}+c_{\mathrm{l}}$. For simplicity, we may set the fraction of shortages backordered $\beta(t_i-u)$ to be approximately equal to $\beta(1)$. Therefore, the expected unit cost of stockout approximately is $\beta(1)c_{\mathrm{s}}+[1-\beta(1)](p-c_{\mathrm{v}}+c_{\mathrm{l}})$. Substituting the above results into Eq. (15) in Teng [19], we obtain an estimate of the number of replenishments for any given p as

$$n=\text{round integer of}\sqrt{\frac{(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)[\beta(1)c_{\mathrm{s}}+[1-\beta(1)](p-c_{\mathrm{v}}+c_{\mathrm{l}})]Q(H\mid p)H}{2c_{\mathrm{f}}[(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)+\beta(1)c_{\mathrm{s}}+[1-\beta(1)](p-c_{\mathrm{v}}+c_{\mathrm{l}})]}}, \tag{13}$$

where $Q(H|p)$ is the accumulative demand during the planning horizon H, i.e.,

$$Q(H\mid p)=\int_0^H A(p)g(t)\,\mathrm{d}t.$$

It is obvious that searching for the optimal number of replenishments by starting with n in (13) instead of $n=1$ will reduce the computational complexity significantly.

We know from Theorem 1 that the optimal replenishment schedule exists and is unique, for any given selling price. Next, we study the conditions under which the optimal selling price also exists and is unique. For any $n, \{s_i\}$ and $\{t_i\}$, the first-order necessary condition for $\mathrm{TP}(p \mid n, \{s_i, \{t_i\})$ to be maximum is

$$\begin{aligned}\frac{\mathrm{dTP}(p \mid n,\{s_i\},\{t_i\})}{\mathrm{d}p} &= \left[A(p) + (p-c_\mathrm{v})\frac{\mathrm{d}A(p)}{\mathrm{d}p}\right]\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}\beta(t_i-t)g(t)\,\mathrm{d}t \\ &\quad + \left[A(p) + (p-c_\mathrm{v})\frac{\mathrm{d}A(p)}{\mathrm{d}p}\right]\sum_{i=1}^{n}\int_{t_i}^{s_i} g(t)\,\mathrm{d}t \\ &\quad - \frac{c_\mathrm{h}+c_\mathrm{v}\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}[\mathrm{e}^{\theta(t-t_i)}-1]g(t)\,\mathrm{d}t\frac{\mathrm{d}A(p)}{\mathrm{d}p} \\ &\quad - c_\mathrm{s}\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}(t_i-t)\beta(t_i-t)g(t)\,\mathrm{d}t\frac{\mathrm{d}A(p)}{\mathrm{d}p} \\ &\quad - c_\mathrm{l}\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}[1-\beta(t_i-t)]g(t)\,\mathrm{d}t\frac{\mathrm{d}A(p)}{\mathrm{d}p} = 0. \end{aligned} \tag{14}$$

From (14), we obtain the following theorem.

Theorem 3. *For any given values of $n, \{s_j\}, \{t_i\}$, if the gross profit is a strictly concave function of p (i.e., $d^2(p-c_v)A(p)/dp^2 < 0$), then*

1. *The solution to (14) satisfies the second-order condition for maximum.*
2. *There exists a unique optimal selling price p^* that satisfies (14).*

Proof. Please see Appendix B for details. □

It is clear from (14) that $\mathrm{dTP}(p \mid n, \{s_i\}, \{t_i\})/\mathrm{d}p = 0$ has a solution if $A(p)+(p-c_\mathrm{v})A'(p)<0$. If the gross profit is a strictly concave function of p, then $A(p)+(p-c_\mathrm{v})A'(p)$, which is the derivative of the concave $(p-c_\mathrm{v})A(p)$, is a strictly decreasing function of p. As a result, the solution of $A(p)+(p-c_\mathrm{v})A'(p)=0$, say p_l, is the lower bound for the optimal selling price p^* such that $\mathrm{dTP}(p \mid n, \{s_i\}, \{t_i\})/\mathrm{d}p = 0$.

Combining the above three theorems, we propose the following algorithm for solving the problem.

Algorithm

Step 1. Start with $j=0$ and the initial trial value of $p_j = p_\mathrm{l}$, which is a solution to $A(p)+(p-c_\mathrm{v})A'(p)=0$.

Step 2. Find the optimal replenishment schedule for a given selling price p_j.

Step 2-1. Choose two initial trial values of n^*, say n as in (13) and $n-1$. Use a standard search method to obtain $\{t_i^*\}$ and $\{s_i^*\}$, and compute the corresponding $\mathrm{TP}(n \mid p_j)$ and $\mathrm{TP}(n-1 \mid p_j)$, respectively.

Step 2-2. If $\mathrm{TP}(n \mid p_j) \leqslant \mathrm{TP}(n-1 \mid p_j)$, then compute $\mathrm{TP}(n-2 \mid p_j), \mathrm{TP}(n-3 \mid p_j), \ldots$, until we find $\mathrm{TP}(k \mid p_j) > \mathrm{TP}(k-1 \mid p_j)$. Set $n^* = k$ and stop.

Step 2-3. If $\mathrm{TP}(n \mid pj) > \mathrm{TP}(n-1 \mid p_j)$, then compute $\mathrm{TP}(n+1 \mid p_j)$, $\mathrm{TP}(n+2 \mid p_j)$, until we find $\mathrm{TP}(k \mid p_j) > \mathrm{TP}(k+1 \mid p_j)$. Set $n^* = k$ and stop.

Step 3. Use the result in Step 2, and then determine the optimal selling price p_{j+1} by (14).

Step 4. If the difference between p_j and p_{j+1} is sufficiently small, set $p^* = p_{j+1}$, then (p^*, n^*) is the optimal solution and stop. Otherwise, set $j = j+1$ and go back to Step 2.

From Theorems 1 and 2, we know that for any given p, there exists a unique optimal solution (n^*, s_i^*, t_i^*) to maximize $\mathrm{TP}(p, n, s_i, t_i)$. Although we could not prove that $TP(p) = TP(n^*, s_i^*, t_i^* \mid p)$ is strictly concave in p, our numerical computational results (as shown in Examples 1 and 2 below) indicate that $\mathrm{TP}(p)$ seems to be strictly concave in p. However, to insure that the local maximum obtained from the proposed algorithm is indeed a global maximum solution, we should repeatedly use the proposed algorithm with distinct starting values of p.

Moreover, the optimal replenishment policy has the following useful properties.

Theorem 4. *For any given n and p, we have*

(a) *If $g(t)$ is increasing with respect to t, then the shortage periods and the inventory periods are getting smaller with respect to the number of replenishment, i.e.,* $t_i - s_{i-1} > t_{i+1} - s_i$ *and* $s_i - t_i > s_{i+1} - t_{i+1}$ *for* $i = 1, 2, \ldots, n-1$.

(b) *If $g(t)$ is decreasing with respect to t, then the shortage periods and the inventory periods are getting larger with respect to the number of replenishment, i.e.,* $t_i - s_{i-1} < t_{i+1} - s_i$ *and* $s_i - t_i < s_{i+1} - t_{i+1}$ *for* $i = 1, 2, \ldots, n-1$.

Proof. The reader can prove it by using the same analogue as in Papachristos and Skouri [15] or Teng et al. [20]. □

5. Numerical examples

To illustrate the results, let us apply the proposed algorithm to solve the following numerical examples.

Example 1. We redo the same example of Wee [24] to see the optimal replenishment policy while considering time-dependent backlogging rate. $f(t,p) = (500 - 0.5p)e^{-0.98t}$, $c_f = 250$, $c_h = 40$, $c_s = 80$, $c_l = 120$, $c_v = 200$, $\theta = 0.08$, $H = 4$, $p_u = 1000$, $\beta(x) = 1/(1 + 10x)$. Solving $A(p) + (p - c_v)A'(p) = 0$ first, we get $p_1 = p_0 = 600$. Then, applying the algorithm, after three iterations, we obtain the optimal replenishment policy as shown in Table 2. The optimal values of p, n and TP are $p^* = 607.6$, $n^* = 5$ and $\mathrm{TP}^* = 77{,}460$, respectively. The total profit obtained here is at least 4.3% higher than that in Wee [24]. This is because Wee [24] assumed that all replenishment intervals are equal. Note that we run the numerical results with different starting values of $p = 500, 510, 520, \ldots, 700$. The numerical results reveal that $TP(p) = TP(n^*,\ s_i^*, t_i^* \mid p)$ is strictly concave in p, as shown in Fig. 2. As a result, we are sure that the local maximum obtained here from the proposed algorithm is indeed the global maximum solution.

Table 2
Optimal replenishment schedule

i	t_i^*	s_i^*
1	0.0033	0.3616
2	0.3656	0.8068
3	0.8118	1.3890
4	1.3956	2.2426
5	2.2527	4.0000

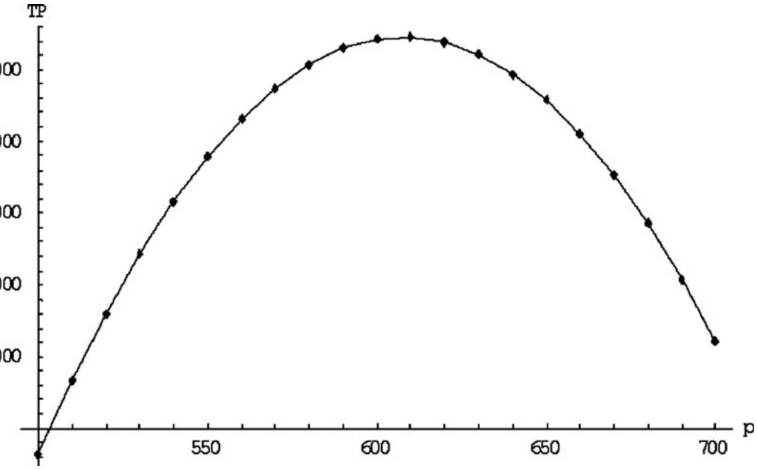

Fig. 2. Graphical representation of $\mathrm{TP}(n^*, s_i^*, t_i^* \mid p)$ in Example 1.

Table 3
Optimal replenishment schedule

i	t_i^*	s_i^*
1	0.2621	0.8799
2	1.1254	1.7121
3	1.9445	2.5056
4	2.7272	3.2666
5	3.4792	4.0000

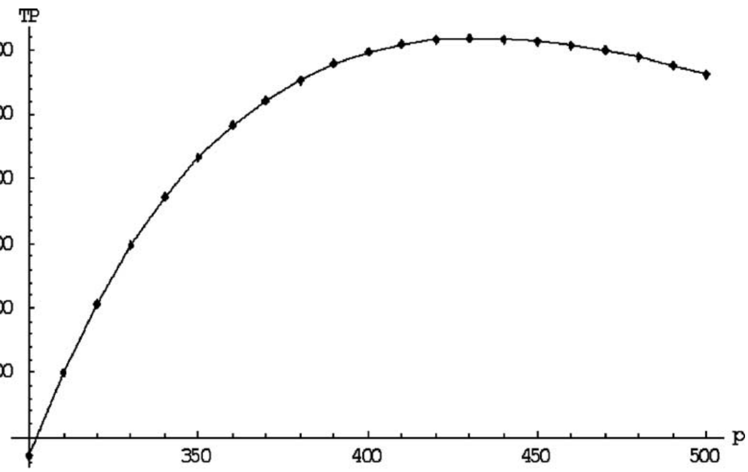

Fig. 3. Graphical representation of $\mathrm{TP}(n^*, s_i^*, t_i^* \mid p)$ in Example 2.

Example 2. $f(t,p)=30{,}000p^{-2}(100+15t)$, $c_{\mathrm{f}}=250$, $c_{\mathrm{h}}=40$, $c_{\mathrm{s}}=80$, $c_{\mathrm{l}}=120$, $c_{\mathrm{v}}=200$, $\theta=0.08$, $H=4$, $p_u=1000$, $\beta(x)=\mathrm{e}^{-0.2x}$. By solving $A(p)+(p-c_{\mathrm{v}})A'(p)=0$, we obtain $p_1=p_0=400$. After two iterations, we have $p^*=430.5$, $n^*=5$ and $\mathrm{TP}^*=17091.1$. The optimal replenishment schedule is shown in Table 3. Note that we run the numerical results with distinct starting values of $p=300,310,320,\ldots,500$. Again, the numerical results indicate that $\mathrm{TP}(p)=\mathrm{TP}(n^*,s_i^*,t_i^*\mid p)$ is strictly concave in p, as shown in Fig. 3. Consequently, we are sure that the local maximum obtained here is indeed the global maximum solution.

6. Conclusion

In this paper, we establish an appropriate model for a retailer to determine its optimal selling price, replenishment number and replenishment schedule. The proposed model allows not only the partial backlogging rate to be related to the waiting time but also a constant deterioration rate. Consequently, the proposed model is in a general framework that includes numerous previous models as special cases. In addition, we propose a simple, effective algorithm to solve the problem with $2+2n$ decision variables. Furthermore, the demand function used in this model is quite general. So it gives some flexibility to cover many demand scenarios.

The proposed model can be extended in several ways. For instance, we may consider the permissible delay in payments. Also, we could extend the deterministic demand function to stochastic fluctuating demand patterns. Finally, we could generalize the model to allow for quantity discounts, inflation and others.

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Appendix A

Proof of the part (a) of Theorem 1. If $f(t)$ is positive and log-concave, after using similar arguments in [5,12,15], we have

$$\begin{aligned}(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)&\left[\mathrm{e}^{\theta(s_i-t_i)}g(s_i)-g(t_i)-\theta\int_{t_i}^{s_i}\mathrm{e}^{\theta(t-t_i)}g(t)\,\mathrm{d}t\right]\\&\leqslant c_{\mathrm{s}}[g(t_i)-\beta(t_i-s_{i-1})g(s_{i-1})-(t_i-s_{i-1})\beta'(t_i-s_{i-1})g(s_{i-1})]\\&\quad+c_{\mathrm{s}}\int_{s_{i-1}}^{t_i}[2\beta'(t_i-t)+(t_i-t)\beta''(t_i-t)]g(t)\,\mathrm{d}t\\&\quad-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\left\{\beta'(0)g(t_i)-\beta'(t_i-s_{i-1})g(s_{i-1})+\int_{s_{i-1}}^{t_i}\beta''(t_i-t)g(t)\,\mathrm{d}t\right\}.\end{aligned}\tag{A.1}$$

Substituting $M_i=s_i-t_i$ and $K_i=t_i-s_{i-1}$ into (10) and (11), and then differentiating them with respect to t_1 yields

$$
\begin{aligned}
&(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\mathrm{e}^{\theta M_i}g(s_i)\frac{\partial M_i}{\partial t_1}+(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\left[\mathrm{e}^{\theta M_i}g(s_i)-g(t_i)-\theta\int_{t_i}^{s_i}\mathrm{e}^{\theta(t-t_i)}g(t)\,\mathrm{d}t\right]\frac{\partial t_i}{\partial t_1}\\
&\quad=c_{\mathrm{s}}\int_{s_{i-1}}^{t_i}[2\beta'(t_i-t)+(t_i-t)\beta''(t_i-t)]g(t)\,\mathrm{d}t\frac{\partial t_i}{\partial t_1}+c_{\mathrm{s}}g(t_i)\frac{\partial t_i}{\partial t_1}\\
&\qquad-c_{\mathrm{s}}[\beta(K_i)+K_i\beta'(K_i)]g(s_{i-1})\frac{\partial t_i}{\partial t_1}-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\int_{s_{i-1}}^{t_i}\beta''(t_i-t)g(t)\,\mathrm{d}t\frac{\partial t_i}{\partial t_1}+(p-c_{\mathrm{v}}+c_{\mathrm{l}})\beta'(K_i)g(s_{i-1})\frac{\partial t_i}{\partial t_1}\\
&\qquad-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\beta'(0)g(t_i)\frac{\partial t_i}{\partial t_1}+\{c_{\mathrm{s}}[\beta(K_i)+K_i\beta'(K_i)]-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\beta'(K_i)\}g(s_{i-1})\frac{\partial K_i}{\partial t_1}
\end{aligned}\tag{A.2}
$$

and

$$
(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\mathrm{e}^{\theta(s_i-t_i)}\frac{\partial M_i}{\partial t_1}=[c_{\mathrm{s}}\beta(K_{i+1})+c_{\mathrm{s}}K_{i+1}\beta'(K_{i+1})-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\beta'(K_{i+1})]\frac{\partial K_{i+1}}{\partial t_1},\tag{A.3}
$$

respectively. It is clear that $\partial K_1/\partial t_1=1$ and, from (A.1) and (A.2), we have $\partial M_1/\partial t_1>0$. Using the fact that $\partial M_1/\partial t_1>0$ and the assumption $c_{\mathrm{s}}\beta(K_{i+1})+c_{\mathrm{s}}H\beta'(K_{i+1})>0$, we can obtain $\partial K_2/\partial t_1>0$ from (A.3). By repeating this procedure mentioned above, we can show that $\partial K_i/\partial t_1>0$ and $\partial M_i/\partial t_1>0$ for $i=1,2,\ldots,n$ from (A.2) and (A.3). Since $s_n(t_1)\equiv s_n=\sum_{i=1}^{n}(M_i+K_i)$ it is easily shown that $\partial s_n(t_1)/\partial t_1=\sum_{i=1}^{n}(\partial M_i/\partial t_1+\partial K_i/\partial t_1)>0$. Furthermore, it is not difficult to check from (10) and (11) that $s_n(0)<H$ and $s_n(H)>H$, thus the Intermediate Value Theorem implies that the root of $s_n(t_1)=H$ is unique. Therefore, the solution to Eqs. (10) and (11) not only exists but is also unique. This completes the proof of Part (a) of Theorem 1. □

Proof the part (b) of Theorem 1. We first obtain the second-order derivatives of the $\mathrm{TP}(\{s_i\},\{t_i\}\mid n,p)$. With (8), (9), (10), (A.1) and the condition $\beta(x)+H\beta'(x)>0$ for all $x\geqslant 0$, we have

$$
\begin{aligned}
\frac{\partial^2\mathrm{TP}}{\partial t_i^2}&\equiv\frac{\partial^2\mathrm{TP}(\{s_i\},\{t_i\}\mid n,p)}{\partial t_i^2}\\
&=-(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\left[\theta\int_{t_i}^{s_i}\mathrm{e}^{\theta(t-t_i)}g(t)\,\mathrm{d}t+g(t_i)\right]A(p)\\
&\quad-c_{\mathrm{s}}\left\{\int_{s_{i-1}}^{t_i}[2\beta'(t_i-t)+(t_i-t)\beta''(t_i-t)]g(t)\,\mathrm{d}t+g(t_i)\right\}A(p)\\
&\quad+(p-c_{\mathrm{v}}+c_{\mathrm{l}})\left[\beta'(0)g(t_i)+\int_{s_{i-1}}^{t_i}\beta''(t_i-t)g(t)\,\mathrm{d}t\right]A(p)\\
&\leqslant-(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\mathrm{e}^{\theta(s_i-t_i)}A(p)g(s_i)-[c_{\mathrm{s}}\beta(t_i-s_{i-1})+c_{\mathrm{s}}(t_i-s_{i-1})\beta'(t_i-s_{i-1})\\
&\quad-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\beta'(t_i-s_{i-1})]A(p)g(s_{i-1})<0,\quad i=1,2,\ldots,n,
\end{aligned}\tag{A.4}
$$

$$
\begin{aligned}
\frac{\partial^2\mathrm{TP}}{\partial s_i^2}&\equiv\frac{\partial^2\mathrm{TP}(\{s_i\},\{t_i\}\mid n,p)}{\partial s_i^2}\\
&=-(c_{\mathrm{h}}+c_{\mathrm{v}}\theta)\mathrm{e}^{\theta(s_i-t_i)}A(p)g(s_i)-[c_{\mathrm{s}}\beta(t_{i+1}-s_i)+c_{\mathrm{s}}(t_{i+1}-s_i)\beta'(t_{i+1}-s_i)\\
&\quad-(p-c_{\mathrm{v}}+c_{\mathrm{l}})\beta'(t_{i+1}-s_i)]A(p)g(s_i)<0,\quad i=1,2,\ldots,n-1,
\end{aligned}\tag{A.5}
$$

$$\frac{\partial^2 \mathrm{TP}}{\partial t_i \partial s_i} \equiv \frac{\partial^2 TP(\{s_i\}, \{t_i\} \mid n, p)}{\partial t_i \partial s_i} = (c_\mathrm{h} + c_\mathrm{v}\theta)\mathrm{e}^{\theta(s_i - t_i)} A(p) g(s_i) > 0, \quad i = 1, 2, \ldots, n, \tag{A.6}$$

and

$$\begin{aligned}\frac{\partial^2 \mathrm{TP}}{\partial s_i \partial t_{i+1}} &\equiv \frac{\partial^2 \mathrm{TP}(\{s_i\}, \{t_i\} \mid n, p)}{\partial s_i \partial t_{i+1}} \\ &= [c_\mathrm{s}\beta(t_{i+1} - s_i) + c_\mathrm{s}(t_{i+1} - s_i)\beta'(t_{i+1} - s_i) - (p - c_\mathrm{v} + c_\mathrm{l})\beta'(t_{i+1} - s_i)]A(p)g(s_i) > 0, \\ &\quad i = 1, 2, \ldots, n-1.\end{aligned} \tag{A.7}$$

From (A.6) and (A.7), Eqs. (A.4) and (A.5) can be rewritten as

$$\frac{\partial^2 \mathrm{TP}}{\partial t_1^2} \leqslant -\frac{\partial^2 \mathrm{TP}}{\partial t_1 \partial s_1} - [c_\mathrm{s}\beta(t_1) + c_\mathrm{s} t_1 \beta'(t_1) - (p - c_\mathrm{v} + c_\mathrm{l})\beta'(t_1)]A(p)g(0) < 0, \tag{A.8}$$

$$\frac{\partial^2 \mathrm{TP}}{\partial t_i^2} \leqslant -\frac{\partial^2 \mathrm{TP}}{\partial t_i \partial s_i} - \frac{\partial^2 \mathrm{TP}}{\partial s_{i-1} \partial t_i} < 0, \quad i = 2, 3 \ldots, n, \tag{A.9}$$

and

$$\frac{\partial^2 \mathrm{TP}}{\partial s_i^2} = -\frac{\partial^2 \mathrm{TP}}{\partial t_i \partial s_i} - \frac{\partial^2 \mathrm{TP}}{\partial s_i \partial t_{i+1}} < 0, \quad i = 1, 2, \ldots, n-1. \tag{A.10}$$

Let $(\{t_i^*\}, \{s_i^*\})$ to be the solutions of (8) and (9), then the Hessian matrix at the stationary points $(\{t_i^*\}, \{s_i^*\})$ is given by

$$\nabla = \begin{bmatrix} \frac{\partial^2 \mathrm{TP}}{\partial t_1^2} & \frac{\partial^2 \mathrm{TP}}{\partial t_1 \partial s_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial^2 \mathrm{TP}}{\partial s_1 \partial t_1} & \frac{\partial^2 TP}{\partial s_1^2} & \frac{\partial^2 \mathrm{TP}}{\partial s_1 \partial t_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2 \mathrm{TP}}{\partial t_2 \partial s_1} & \frac{\partial^2 \mathrm{TP}}{\partial t_2^2} & \frac{\partial^2 \mathrm{TP}}{\partial t_2 \partial s_2} & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 \mathrm{TP}}{\partial t_{n-1} \partial s_{n-2}} & \frac{\partial^2 \mathrm{TP}}{\partial t_{n-1}^2} & \frac{\partial^2 \mathrm{TP}}{\partial t_{n-1} \partial s_{n-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 \mathrm{TP}}{\partial s_{n-1} \partial t_{n-1}} & \frac{\partial^2 \mathrm{TP}}{\partial s_{n-1}^2} & \frac{\partial^2 \mathrm{TP}}{\partial s_{n-1} \partial t_n} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 \mathrm{TP}}{\partial t_n \partial s_{n-1}} & \frac{\partial^2 \mathrm{TP}}{\partial t_n^2} \end{bmatrix}. \tag{A.11}$$

Based on the inequalities (A.9) and (A.10), and the results by Balkhi [3] and Stewart [18], we know that the tridiagonal matrix in (A.11) is negative definite. □

Appendix B

Proof of Theorem 3. Taking the second derivative of $\mathrm{TP}(p\,|\,n,\{s_i\},\{t_i\})$ with respect to p, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2\mathrm{TP}(p\mid n,\{s_i\},\{t_i\})}{\mathrm{d}p^2} &= \left[2\frac{\mathrm{d}A(p)}{\mathrm{d}p}+(p-c_{\mathrm{v}})\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2}\right]\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}\beta(t_i-t)g(t)\,\mathrm{d}t \\ &\quad+\left[2\frac{\mathrm{d}A(p)}{\mathrm{d}p}+(p-c_{\mathrm{v}})\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2}\right]\sum_{i=1}^{n}\int_{t_i}^{s_i}g(t)\,\mathrm{d}t \\ &\quad-\frac{c_{\mathrm{h}}+c_{\mathrm{v}}\theta}{\theta}\sum_{i=1}^{n}\int_{t_i}^{s_i}[\mathrm{e}^{\theta(t-t_i)}-1]g(t)\,\mathrm{d}t\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2} \\ &\quad-c_{\mathrm{s}}\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}(t_i-t)\beta(t_i-t)g(t)\,\mathrm{d}t\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2} \\ &\quad-c_{\mathrm{l}}\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}[1-\beta(t_i-t)]g(t)\,\mathrm{d}t\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2}.\end{aligned} \tag{B.1}$$

Since

$$\frac{\mathrm{d}^2(p-c_{\mathrm{v}})A(p)}{\mathrm{d}p^2}=2\frac{\mathrm{d}A(p)}{\mathrm{d}p}+(p-c_{\mathrm{v}})\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2}<0\quad\text{and}\quad\frac{\mathrm{d}^2A(p)}{\mathrm{d}p^2}>0$$

it is clear that $\mathrm{d}^2\mathrm{TP}(p|n,\{s_i\},\{t_i\})/\mathrm{d}p^2<0$. Consequently, $\mathrm{TP}(p\mid n,\{s_i\},\{t_i\})$ is a strictly concave function of p. For any given values of $n,\{s_i\},\{t_i\},\mathrm{TP}(p\mid n,\{s_i\},\{t_i\})$ is a continuous function of p on a compact set $[0,p_u]$, where p_u is an extremely large number. It is obvious from (B.1) that $\mathrm{TP}(p\mid n,\{s_i\},\{t_i\})$ is not optimal if $p=0$ or p_u. As a result, the optimal p must be an interior point between 0 and p_u. This implies that Eq. (14) has at least one solution. Next, since $\mathrm{TP}(p|n,\{s_i\},\{t_i\})$ is a strictly concave function of p, we know that there exists a unique interior value of p that maximizes $\mathrm{TP}(p|n,\{s_i\},\{t_i\})$. This completes the proof. □

References

[1] P.L. Abad, Optimal pricing and lot sizing under conditions of perishability and partial backordering, Management Science 42 (1996) 1093–1104.

[2] P.L. Abad, Optimal price and order size for a retailer under partial backordering, Computers and Operations Research 28 (2001) 53–65.

[3] Z.T. Balkhi, On a finite horizon production lot size inventory model for deteriorating items: An optimal solution, European Journal of Operational Research 132 (2001) 210–223.

[4] L.C. Barbosa, M. Friedman, Deterministic inventory lot size models–a general root law, Management Science 24 (1978) 819–826.

[5] H.J. Chang, C.Y. Dye, An EOQ model for deteriorating items with time varying demand and partial backlogging, Journal of the Operational Research Society 50 (1999) 1176–1182.

[6] U. Dave, A deterministic lot-size inventory model with shortages and a linear trend in demand, Naval Research Logistics 36 (1989) 507–514.

[7] U. Dave, L.K. Patel, (T, S_i) policy inventory model for deteriorating items with time proportional demand, Journal of the Operational Research Society 32 (1) (1981) 137–142.

[8] W.A. Donaldson, Inventory replenishment policy for a linear trend in demand: An analytical solution, Operational Research Quarterly 28 (1977) 663–670.

[9] M.F. Friedman, Inventory lot-size models with general time-dependent demand and carrying cost function, INFORM 20 (1982) 157–167.

[10] S.K. Goyal, B.C. Giri, Recent trends in modeling of deteriorating inventory, European Journal of Operational Research 134 (2001) 1–16.

[11] S.K. Goyal, D. Morin, F. Nebebe, The finite horizon trended inventory replenishment problem with shortages, Journal of the Operational Research Society 43 (1992) 1173–1178.

[12] M. Hariga, Optimal EOQ models for deteriorating items with time-varying demand, Journal of the Operational Research Society 47 (1996) 1228–1246.
[13] M.A. Hariga, S.K. Goyal, An alternative procedure for determining the optimal policy for an inventory item having linear trend in demand, Journal of the Operational Research Society 46 (1995) 521–527.
[14] R.J. Henery, Inventory replenishment policy for increasing demand, Journal of the Operational Research Society 30 (1979) 611–617.
[15] S. Papachristos, K. Skouri, An optimal replenishment policy for deteriorating items with time-varying demand and partial-exponential type-backlogging, Operations Research Letters 27 (2000) 175–184.
[16] M. Resh, M. Friedman, L.C. Barbosa, On a general solution of the deterministic lot size problem with time-proportional demand, Operations Research 24 (1976) 718–725.
[17] R.S. Sachan, On (T, S_i) policy inventory model for deteriorating items with time proportional demand, Journal of the Operational Research Society 35 (1984) 1013–1019.
[18] C.W. Stewart, Introduction to Matrix Computations, Academic Press, New York, 1973.
[19] J.T. Teng, A deterministic replenishment model with linear trend in demand, Operations Research Letters 19 (1996) 33–41.
[20] J.T. Teng, H.J. Chang, C.Y. Dye, C.H. Hung, An optimal replenishment policy for deteriorating items with time-varying demand and partial backlogging, Operations Research Letters 30 (2002) 387–393.
[21] J.T. Teng, M.S. Chern, H.L. Yang, An optimal recursive method for various inventory replenishment models with increasing demand and shortages, Naval Research Logistics 44 (1997) 791–806.
[22] J.T. Teng, M.S. Chern, H.L. Yang, Y.J. Wang, Deterministic lot-size inventory models with shortages and deterioration for fluctuating demand, Operations Research Letters 24 (1999) 65–72.
[23] H.L. Yang, J.T. Teng, M.S. Chern, Deterministic inventory lot-size models under inflation with shortages and deterioration for fluctuating demand, Naval Research Logistics 48 (2001) 144–158.
[24] H.M. Wee, Joint pricing and replenishment policy for deteriorating inventory with declining market, International Journal of Production Economics 40 (1995) 163–171.